Measuring Consistency of Fuzzy Logic Theories

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Introduction

- Inconsistency naturally appears when fusing information from different sources.
- If we understand inconsistency in fuzzy logic theories as the absence of models, then:
  - Some of the most important properties of inconsistency are kept in the fuzzy setting.
  - But we also lose degrees, which are the soul of fuzzy logic.
- A number of papers have focused on measuring the degree of inconsistency of a set of fuzzy rules, and a number of different inconsistency indices have been introduced.
- This suggests that there is not a consensus on how to interpret inconsistency in a fuzzy system. We focus on, under of point of view, the most natural way to define inconsistency in a logic theory, namely, the absence of models.
- We define here two measures of consistency that belong purely to the fuzzy paradigm.
- Both are generalizations, namely, coincide with the crisp notion of consistency when the underlying set of truth values is $\{0, 1\}$.

Outline

- Preliminary definitions.
- Inconsistency.
- Contradiction.
- Measure of consistency for formulas.
- Measure of consistency for fuzzy logic theories.
- Conclusions and future work.

Preliminary definitions

Residuated lattices

A residuated lattice, which is a tuple $(L, \leq, *, \rightarrow, 0, 1)$ such that:

- $(L, \leq)$ is a complete bounded lattice, with top and bottom element 1 and 0, respectively.
- $(L, *, 1)$ is a commutative monoid with unit element 1.
- $(*, \rightarrow)$ forms an adjoint pair, i.e.

$$z \leq (y \rightarrow x) \quad \text{if and only if} \quad y * z \leq x$$

Preliminary definitions

Residuated lattices

We restrict our attention to residuated lattices defined on the unit interval $[0, 1]$. In the examples, we will often use some of the main residuated lattices on $[0, 1]$ given by the following operators:

- Gödel residuated lattice $([0, 1], \leq, *_G, \rightarrow_G, 0, 1)$

$$x *_G y = \min\{x, y\} \qquad x \rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$$

- Product residuated lattice $([0, 1], \leq, *_P, \rightarrow_P, 0, 1)$

$$x *_P y = x \cdot y \qquad x \rightarrow_P y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise.} \end{cases}$$

- Łukasiewicz residuated lattice $([0, 1], \leq, *_Ł, \rightarrow_Ł, 0, 1)$

$$x *_Ł y = \max\{x + y - 1, 0\} \qquad x \rightarrow_Ł y = \min\{1 + y - x, 1\}$$

Preliminary definitions

Syntax of fuzzy logic theories

The formulas of the fuzzy logic based on a residuated lattice $(L, \leq, *, \rightarrow, 0, 1)$ are given by the following inductive definition based on a primitive set of propositional symbols Π:

- every propositional symbol in Π is a well-formed formula;
- every element of L is a well-formed formula;
- if ϕ and ψ are well-formed formula then:
  - $\varphi * \psi$ is a well-formed formula;
  - $\varphi \rightarrow \psi$ is a well-formed formula;
  - $\varphi \wedge \psi$ is a well-formed formula;
  - $\varphi \vee \psi$ is a well-formed formula;
  - $\neg\varphi$ is a well-formed formula.

Preliminary definitions

Semantics of fuzzy logic theories

Let $(L, \leq, *, \rightarrow, 0, 1)$ be a residuated lattice. An interpretation is a mapping $I \colon \Pi \rightarrow L$.

The domain of an interpretation I can be extended inductively to any well-formed formula as follows, where φ and ψ are well-formed formulas. :

- $I(a) = a$ for all $a \in L$;
- $I(\varphi * \psi) = I(\varphi) * I(\psi)$;
- $I(\varphi \rightarrow \psi) = I(\varphi) \rightarrow I(\psi)$;
- $I(\varphi \wedge \psi) = \inf\{I(\varphi), I(\psi)\}$;
- $I(\varphi \vee \psi) = \sup\{I(\varphi), I(\psi)\}$;
- $I(\neg\varphi) = I(\varphi) \rightarrow 0$;

The definition of theory, model of a theory and logical consequence in a theory are given as follows:
Let $(L, \leq, *, \rightarrow, 0, 1)$ be a residuated lattice.

- A fuzzy logic theory Γ is a set of well-formed formulas.
- A model of Γ is an interpretation M such that $M(\psi) = 1$ for all $\psi \in \Gamma$.
- We say that a formula ψ is a consequence of Γ (denoted by $\Gamma \vDash \psi$) if $M(\psi) = 1$ for all model M of Γ.

Inconsistency of a Logic Theory

There are several ways to define an inconsistent logic theory:

- **explosive reasoning**: if entails every formula can be inferred from it.
- **inference of contradictory set**: if a contradictory set of formulas can inferred. (the most usual example is the inference of two opposite literals p and $\neg p$)
- **the inference of falsehood**: if the formula $\bot$ is a consequence of it.
- **trivial reasoning**: if we can infer a formula such that is not a tautology and none of its propositional symbols appear in the theory.
- **lack of models**: if it has no models.

Inconsistency of a fuzzy logic theory in the strong sense

Definition

A fuzzy logic theory Γ is said to be inconsistent if it has no models.

The previous definition of inconsistency is equivalent to the one related to *explosive reasoning* and *trivial reasoning*.

Theorem

Let Γ be a fuzzy logic theory. Then the following statements are equivalent:

- Γ *is inconsistent.*
- $\Gamma \models \psi$ *for all formula* ψ.
- *There exists a formula* ψ *whose symbols do not appear in* Γ *such that* $\Gamma \models \psi$ *and* $\varnothing \not\models \psi$.

What about contradictory formulas?

There are different ways to define contradictory formulas in fuzzy logic according to this general idea:

$$\textit{Contradiction} = \textit{combination of statements which are opposed to one another}$$

We can define contradiction directly by means of inconsistency as follows:

Definition

Let $(L, \leq, *, \rightarrow)$ be a residuated lattice. We say that a set $\{\psi_i\}_{i \in \mathbb{N}}$ of formulas is contradictory if is inconsistent as a logic theory.

What about contradictory formulas?

A first example

It is worth remarking that contradiction depends on the underlying residuated structure used in the semantics.

Example

- Let us consider the formulas $0.5 \to p$ and $0.5 \to \neg p$.
- If we analyze the theory $\Gamma = \{0.5 \to p\ ;\ 0.5 \to \neg p\}$ in the product logic then we have that Γ is inconsistent and therefore the formulas $0.5 \to p$ and $0.5 \to \neg p$ are contradictory.
- On the other hand, if we analyze Γ under the Łukasiewicz logic we have that the interpretation M given by $M(p) = 0.5$ is a model of Γ. As a result, $0.5 \to p$ and $0.5 \to \neg p$ are not contradictory in Łukasiewicz logic.

What about contradictory formulas?

Considering the latter definition, we reach the equivalences between the five "classical" ways to define inconsistency in fuzzy logic theories.

Theorem

Let Γ be a fuzzy logic theory. Then the following statements are equivalent

- Γ *is inconsistent.*
- *There exists a contradictory set of formulas* $\{\psi_i\}_{i \in \mathbb{N}}$ *such that* $\Gamma \models \psi_i$ *for all* $i \in \mathbb{N}$.
- *There exists a contradictory formula* ψ *such that* $\Gamma \models \psi$.

The following definition is used to determine an upper bound for the truth value of the formula

Definition

Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and consider $\alpha \in L$.

- A formula ψ is said to be α-feasible w.r.t. Γ if $\Gamma \models \psi \rightarrow \alpha$.
- If $\Gamma = \emptyset$ we say that ψ is just α-feasible.

- The value α is an element in the residuated lattice that determines an upper bound for the truth-value of the formula.
- A 0-feasible formula with respect to a logic theory Γ is a formula that has truth-degree 0 for all model of Γ whereas every formula is 1-feasible.
- The value α can be understood as a degree of compatibility.

Example

- Let us consider the Łukasiewicz residuated lattice $([0,1], \leq, *_{Ł}, \rightarrow_{Ł}, 0, 1)$ and the following two formulas: $\bot$ and $p \wedge \neg p$.
- Both formulas are contradictory, since none of them has a model; i.e., for all interpretation I we have that $I(\bot) \neq 1$ and $I(p \wedge \neg p) \neq 1$.
- However, the inherent contradiction of the former formula can be considered stronger than in the latter, since for all interpretation I we have that $I(\bot) = 0$, which means it is in all cases completely false (0-feasible), whereas there are interpretations I such that $I(p \wedge \neg p) = 0.5$, which means that $p \wedge \neg p$ maybe be half true (at least 0.5-feasible).

Measure of consistency for formulas

α-feasible formula

It is easy to check that an α-feasible formula with $\alpha < 1$ is inconsistent.

Therefore, the notion of α-feasibility aims at determining how inconsistent a formula is.

Proposition

Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$. A formula ψ is α-feasible w.r.t Γ if and only if $M(\psi) \leq \alpha$ for all model M of Γ.

The notion of α-feasibility is related to the notion of inconsistency as follows.

Proposition

Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and consider $\alpha \in L$. If ψ is α-feasible w.r.t Γ then, $\Gamma \cup \{\beta \rightarrow \psi\}$ is inconsistent for all $\beta > \alpha$.

We can obtain the following proposition which shows a certain monotonicity with respect to the value α in the notion of α-feasibility.

Proposition

Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$. If a formula ψ is α-feasible w.r.t Γ then, ψ is β-feasible w.r.t Γ for all $\beta \geq \alpha$.

Measure of consistency for formulas

Defining the measure

Therefore, we can consider the value α in the notion of α-feasibility as opposed to a degree of contradiction and inconsistency.

In fact, by using the previous propositions, we can define the following measure of consistency.

Definition

Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and let ψ be a well-formed formula. The degree of consistency of ψ with respect to Γ is defined as the value:

$$\mathsf{Mc}(\psi, \Gamma) = \min\{\alpha \mid \psi \text{ is } \alpha\text{-}\underline{\text{feasible}} \text{ w.r.t } \Gamma\}$$

Theorem (Properties of Mc)

Let Γ *and* Γ' *be fuzzy logic theories defined on a residuated lattice* $(L, \leq, *, \rightarrow)$ *and let* ψ *and* ϕ *be well-formed formulas. Then:*

- $\mathrm{Mc}(\psi, \Gamma) \geq \mathrm{Mc}(\psi, \Gamma \cup \Gamma')$ *; i.e., more formulas in* Γ *can only reduce consistency of* ψ.
- $\mathrm{Mc}(\psi \wedge \varphi, \Gamma) \leq \inf\{\mathrm{Mc}(\psi, \Gamma), \mathrm{Mc}(\varphi, \Gamma)\}$;
- *If* $\Gamma \vDash \varphi$, *then* $\mathrm{Mc}(\psi * \varphi, \Gamma) = \mathrm{Mc}(\psi, \Gamma)$;
- *If* $\Gamma \vDash \varphi$, *then* $\mathrm{Mc}(\psi \wedge \varphi, \Gamma) = \mathrm{Mc}(\psi, \Gamma)$;
- *If* $\Gamma \cup \{\psi\}$ *is consistent then,* $\mathrm{Mc}(\psi, \Gamma) = 1$.

Measure of consistency for formulas

More properties

Theorem (Properties of Mc, continuation)

- If Γ is consistent and $\Gamma \vDash \psi$ then, $\mathrm{Mc}(\psi, \Gamma) = 1$;
- If $\mathrm{Mc}(\psi, \Gamma) \neq 1$ then, $\Gamma \cup \{\psi\}$ is inconsistent;
- If Γ is inconsistent then, $\mathrm{Mc}(\psi, \Gamma) = 0$ for all formula ψ;
- $\mathrm{Mc}(\top, \Gamma) = 1$ if and only if Γ is consistent;
- $\mathrm{Mc}(\bot, \Gamma) = 0$.

Proposition

Let $(L, \leq, *, \rightarrow)$ be a residuated lattice with L finite and totally ordered, let ψ be a well-formed formula and let Γ be a logic theory. Then $\mathrm{Mc}(\psi, \Gamma) = 1$ if and only if $\Gamma \cup \{\psi\}$ is consistent.

Measure of consistency for formulas

A detailed example: preliminary observations

Let us consider the theory $\Gamma = \{p \to q \;,\; q \to 0.5\}$, and let us analyze the inconsistency of $0.7 \to p$ w.r.t. Γ in each residuated lattice (Gödel, product, and Łukasiewicz).

1. In any of the three cases, the interpretation M_0 defined by $M_0(p) = 0$ and $M_0(q) = 0.5$ is a model of Γ, hence Γ is consistent.
2. Every model M of Γ (independently of the underlying lattice considered) satisfies that $M(p) \leq M(q)$ and $M(q) \leq 0.5$; and as a result, $M(p) \leq 0.5$ as well.
3. Note also that $M_1(p) = M_1(q) = 0.5$ is a model of Γ.
4. In the three residuated lattices the formula $0.7 \to p$ is contradictory w.r.t. Γ since $\Gamma \cup \{0.7 \to p\}$ is inconsistent.
   This is because if M is a model of $\{p \to q \;,\; q \to 0.5 \;,\; 0.7 \to p\}$, then necessarily M must satisfy the inequalities $M(p) \leq M(q)$, $M(q) \leq 0.5$ and $M(p) \geq 0.7$; which is impossible.

Measure of consistency for formulas

A detailed example

Example (Case of GÖDEL LOGIC)

We have to determine the minimum $\alpha \in [0,1]$ such that $0.7 \to p$ is α-feasible w.r.t. Γ; or in other words, we have to determine the set of those $\alpha \in [0,1)$ such that $M\big((0.7 \to p) \to \alpha\big) = 1$ for all model of Γ. Note that in Gödel logic we have:

$$M\big((0.7 \to p) \to \alpha\big) = 1 \Leftrightarrow M(0.7 \to p) \leq \alpha \Leftrightarrow M(p) \leq \alpha \text{ and } M(p) \leq 0.7 \Leftrightarrow M(p) \leq \alpha$$

- By $M(p) \leq 0.5$, we can assert that $0.7 \to p$ is α-feasible w.r.t. Γ for all $\alpha \geq 0.5$.
- Now, by the remarks above, we know that there exists a model such that $M(p) = 0.5$.
- As a result, we can conclude that if $\alpha < 0.5$, then $0.7 \to p$ is not α-feasible w.r.t. Γ since the inequality $M(p) \leq \alpha$ does not hold for the mentioned model.
- Therefore, the measure of consistency is given by the minimum of the interval $[0.5, 1]$, that is $\mathrm{Mc}(0.7 \to p, \Gamma) = 0.5$.

Measure of consistency for formulas

A detailed example

Example (Case of PRODUCT LOGIC)

We have to study the equality $M\big((0.7 \to p) \to \alpha\big) = 1$ with $\alpha \in [0,1)$. In product logic, we have

$$M\big((0.7 \to p) \to \alpha\big) = 1 \Leftrightarrow M(0.7 \to p) \leq \alpha \Leftrightarrow \frac{M(p)}{0.7} \leq \alpha \text{ and } M(p) \leq 0.7 \Leftrightarrow M(p) \leq \alpha \cdot 0.7$$

- In the last equivalence we have used that for all model M we have that $M(p) \leq 0.5$.
- By following a similar reasoning than for the Gödel logic, we can conclude that $0.7 \to p$ is α-feasible if and only if $\alpha \geq \frac{5}{7}$.
- As a result, under the product residuated lattice we have $\mathrm{Mc}(0.7 \to p, \Gamma) = \frac{5}{7}$.

Measure of consistency for formulas

A detailed example

Example (Case of ŁUKASIEWICZ LOGIC)

In this case, given a model M of Γ we have the equivalences:

$$M\big((0.7 \to p\big) \to \alpha) = 1 \Leftrightarrow M(0.7 \to p) \leq \alpha \Leftrightarrow \min\{0.3 + M(p), 1\} \leq \alpha$$

Since $M(p) \leq 0.5$ for all model of Γ and there is a model M of Γ such that $M(p) = 0.5$, the least value of α we can choose to guarantee $\Gamma \models (0.7 \to p) \to \alpha$ has to satisfy $0.3 + 0.5 = 0.8 \leq \alpha$; so in Łukasiewicz logic $\mathrm{Mc}(0.7 \to p, \Gamma) = 0.8$.

Measure of consistency for Fuzzy Logic Theories

Definition

Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$, then we define the measure of consistency $\mathrm{Mc}^*(\Gamma)$ as:

$$\mathrm{Mc}^*(\Gamma) = \sup\left\{ \mathrm{Mc}\left(\bigwedge_{\psi_i \in \Gamma \setminus \Gamma^*} \psi_i \ , \Gamma^* \right) \mid \Gamma^* \subseteq \Gamma \text{ is consistent } \right\}.$$

Theorem

Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$, then:

$$\mathrm{Mc}^*(\Gamma) = \mathrm{Mc}\left(\bigwedge_{\psi_i \in \Gamma} \psi_i \ , \varnothing \right).$$

As a direct consequence of the previous theorem, we have the following properties:

Corollary

Let Γ *and* Γ' *be fuzzy logic theories defined on a residuated lattice* $(L, \leq, *, \rightarrow)$*, then:*

- $\mathrm{Mc}^*(\Gamma) \geq \mathrm{Mc}^*(\Gamma \cup \Gamma')$;
- *If* Γ *is consistent then,* $\mathrm{Mc}^*(\Gamma) = 1$;
- *If* $\mathrm{Mc}^*(\Gamma) \neq 1$ *then,* Γ *is inconsistent;*
- *If* L *is finite and totally ordered, then* $\mathrm{Mc}^*(\Gamma) = 1$ *implies* Γ *is consistent.*

Example (On Gödel logic)

Consider $\Gamma = \{\psi_1, \psi_2, \psi_3\}$ where $\psi_1 = p \to 0.4$, $\psi_2 = q \to p$ and $\psi_3 = 0.5 \to q$.
It is not difficult to check that Γ is inconsistent.
Let us compute the minimum of the set $\{\alpha \mid \psi_1 \wedge \psi_2 \wedge \psi_3$ is α-feasible$\}$; that is, the minimum $\alpha \in [0,1]$ such that $\varnothing \models (\psi_1 \wedge \psi_2 \wedge \psi_3) \to \alpha$.
Let us consider $\alpha = 0.4$ and show that $\varnothing \models (\psi_1 \wedge \psi_2 \wedge \psi_3) \to 0.4$. This amounts to prove that $I(\psi_1 \wedge \psi_2 \wedge \psi_3) \leq 0.4$ for all I, so let us consider an interpretation I and reason by cases:

- If $I(p) > 0.4$, then $I(\psi_1) = I(p \to 0.4) = 0.4$. That implies that $I(\psi_1 \wedge \psi_2 \wedge \psi_3) \leq 0.4$.
- If $I(p) \leq 0.4$ and $I(p) < I(q)$ then $I(\psi_2) = I(q \to p) = I(p) \leq 0.4$ and as a result $I(\psi_1 \wedge \psi_2 \wedge \psi_3) \leq 0.4$.
- If $I(p) \leq 0.4$ and $I(q) \leq I(p)$ then, necessarily $I(q) \leq 0.4$. Therefore, $I(\psi_3) = I(0.5 \to q) = I(q) \leq 0.4$, and then $I(\psi_1 \wedge \psi_2 \wedge \psi_3) \leq 0.4$.

as a result, we have that $\psi_1 \wedge \psi_2 \wedge \psi_3$ is 0.4-feasible.

Example (On Gödel logic, continued)

Now, let us prove that if $\alpha < 0.4$, then $\varnothing \not\models (\psi_1 \wedge \psi_2 \wedge \psi_3) \to \alpha$; for this we have just to provide an interpretation such that $I((\psi_1 \wedge \psi_2 \wedge \psi_3) \to \alpha) \neq 1$. Let us consider the interpretation I given by $I(p) = I(q) = 0.5$. Then, we have:

- $I(\psi_1) = I(p) \to 0.4 = 0.5 \to 0.4 = 0.4$
- $I(\psi_2) = I(q) \to I(p) = 0.5 \to 0.5 = 1$
- $I(\psi_3) = 0.5 \to I(q) = 0.5 \to 0.5 = 1$

therefore $I(\psi_1 \wedge \psi_2 \wedge \psi_3) = 0.4 \not\leq \alpha$ and $I((\psi_1 \wedge \psi_2 \wedge \psi_3) \to \alpha) = \alpha \neq 1$.
As a conclusion, we can say that $\{\alpha \mid \psi_1 \wedge \psi_2 \wedge \psi_3$ is α-feasible$\} = [0.4, 1]$ and then $\mathrm{Mc}^*(\Gamma) = 0.4$.

Conclusions

We have presented two different measures of consistency.

- The first one measures how much compatible a formula is with respect to a given theory in the sense:
  - the closer to 0, the more inconsistent;
  - and the closer to 1, the more consistent.
- The second measure determines a degree of consistency of a logic theory by means of consistent subtheories.

Both definitions coincide with the standard notion of consistency when restricted to crisp logic.

Both definitions satisfy convenient properties in order to be considered measures of consistency.

Future Works

There are two main lines of future research.

- On the one hand it is convenient to keep digging up some measures of inconsistency in fuzzy paradigms.
- On the other hand, it is interesting to find out an application of the measures of consistence. For instance, we think they can be used to deal with contradictions in databases obtained from fails or system errors.
- For more information and details, this talk has been based on the following paper

N. Madrid and M. Ojeda-Aciego.
A measure of consistency for fuzzy logic theories.
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