# Extremes of Gaussian RF's, Asymptotic Constants, Spectral Tail RF's, Tail Measures & Cluster RF's

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# Outline

- Extremes of $\mathbb{R}^d$-valued GP's
- Asymptotic constants
- Shift-invariant classes of rf's
- Spectral tail rf's
- Cluster rf's
- Applications

# Extremal behaviour of Gaussian RV's

Consider $\boldsymbol{X} = (X_1, \ldots, X_d) \sim N(\mathbf{0}, \Sigma)$ with $\Sigma$ non-singular.

**Q1**: $\mathbb{P}\{X_i > b_i u, \forall\ i = 1, \ldots, d\}$ for $u$ large?

**A1**: Solve first the quadratic optimisation problem

$$\Pi_\Sigma(\boldsymbol{b}): \qquad \text{minimise}: \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x}, \quad \boldsymbol{x} \ge \boldsymbol{b} \tag{1}$$

$$\ln \mathbb{P}\{\boldsymbol{X} > \boldsymbol{b} u\} \sim -u^2 \frac{\tau}{2}, \quad \tau = \inf_{\boldsymbol{x} \ge \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} \tag{2}$$

**Example: [2-dim case $\boldsymbol{b} = (b, b)^\top, b > 0$]**

$\tau = \boldsymbol{b}^\top \Sigma^{-1} \boldsymbol{b}$, however for general $\boldsymbol{b}$

$$argmin_{\boldsymbol{x} \ge \boldsymbol{b}} \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x} =: \widetilde{\boldsymbol{b}} \neq \boldsymbol{b}$$

# Exact asymptotics: Loss of dimensions phenomenon

The exact asymptotics is given for $u \to \infty$ by (see e.g., [1])

$$\mathbb{P}\{\boldsymbol{X} > \boldsymbol{b}u\} \sim \mathbf{c}\mathbb{P}\{\boldsymbol{X}_I > \boldsymbol{b}_I u\} \sim c_\star u^{-|I|}\varphi(\boldsymbol{b}u) \quad (3)$$

for some unique index set $I \subset \{1, \ldots, d\}$ and $\varphi$ the pdf of $\boldsymbol{X}_I$.

**Example: [$(X_1, X_2)$ with $N(0,1)$ marginals and $\rho \in (-1,1)$]**

If $\boldsymbol{b} = (1, b)^\top$ with $b \leq \rho$, then $I = \{1\}$ and

$$\mathbb{P}\{X_1 > u, X_2 > bu\} \sim c\mathbb{P}\{X_1 > u\}$$

with $\mathbf{c} = 1/2$ when $b = \rho$ and $\mathbf{c} = 1$, otherwise.

# Extremes of stationary GP's

$X(t), t \geq 0$ is a centered, **stationary GP** with continuous paths, unit variance & correlation $r(t) < 1, \forall t > 0$ satisfying

$$1 - r(t) \sim |t|^{\alpha}, \quad t \to 0, \quad \alpha \in (0, 2]$$

Pickands '69 [2] showed that

$$\mathbb{P}\left\{ \sup_{t \in [0,T]} X(t) > u \right\} \sim \mathcal{H} T u^{2/\alpha} \mathbb{P}\left\{ X(0) > u \right\} \tag{4}$$

where $\mathcal{H}$ is the Pickands constant,
$\sim$ means asymp equivalence.

# Extremes of vector-valued GP's

$\boldsymbol{X}(t) = (X_1(t), \ldots, X_d(t))^\top$, $t \in [0, T]$ is a centered **GP** with continuous paths

$\boldsymbol{b} = (b_1, \ldots, b_d)^\top \in \mathbb{R}^d \setminus (-\infty, 0]^d$

**Q2**: How to approximate

$$\mathbb{P}\{\exists t \in [0, T] : \ \boldsymbol{X}(t) > \boldsymbol{b}u\}$$

as $u \to \infty$?

In our notation

$$\boldsymbol{x} > \boldsymbol{y} \Longleftrightarrow x_i > y_i, \quad 1 \le i \le d$$

## log-asymptotics

Define $\Sigma(t) = \mathbb{E}\{\boldsymbol{X}(t)\boldsymbol{X}(t)^\top\}$ and determine

$$\tau = \inf_{t\in[0,T]} \min_{\boldsymbol{x}\ge\boldsymbol{b}} \boldsymbol{x}^\top (\Sigma(t))^{-1}\boldsymbol{x} \tag{5}$$

then similarly to (6) (Debicki et al. '10, [3], Debicki et al. '24+)

$$\ln \mathbb{P}\{\exists t\in[0,T]:\ \boldsymbol{X}(t) > \boldsymbol{b}u\} \sim -u^2\frac{\tau}{2} \tag{6}$$

Recent results dealing with **exact asymptotics** in [1, 4, 5].

# Pickands constants

Given the fractional Brownian motion $B(t), t \in \mathbb{R}$ with

$$Cov(B(s), B(t)) = |t|^{\alpha} + |s|^{\alpha} - |t-s|^{\alpha}, \quad \alpha \in (0, 2]$$

define with $\widetilde{B}(t) = B(t) - Var(B(t))/2$ the Pickands constants

$$\mathcal{H}^{\delta} = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left\{ \sup_{t \in [0,T] \cap \delta\mathbb{Z}} e^{\widetilde{B}(t)} \right\}$$

where $\delta\mathbb{Z} =: \mathbb{R}$ if $\delta = 0$.

**Pickands '69 proved** $\mathcal{H}^{\delta} \in (0, \infty)$ **and**

$$\mathcal{H} = \lim_{\delta \downarrow 0} \mathcal{H}^{\delta} \tag{7}$$

# Exact values & simulations

If $\alpha = 2$ in view of Debicki & H. '17, [6]

$$\mathcal{H}^{\delta} = \frac{1}{\delta}[\Phi(\delta/\sqrt{2}) - \Phi(-\delta/\sqrt{2})]$$

with $\Phi$ the **standard Gaussian df**.
Letting $\delta \to 0$ yields

$$\mathcal{H} = \sqrt{2}\Phi'(0) = 1/\sqrt{\pi}$$

If $\alpha = 1$, the case of Brownian motion, $\mathcal{H} = 1$.

**Q3**: **How to calculate/simulate those constants?**

# Berman representation

**A3**: For simulation a formula in terms of expectations is useful.
In view of Berman '92, [7]

$$\mathcal{H} = \mathbb{E}\left\{ \frac{1}{\int_{t\in\mathbb{R}} \mathbb{I}\{\widetilde{B}(t) + E > \mathbf{0}\} dt} \right\} \tag{8}$$

with $E$ a unit exponential rv independent of $\widetilde{B}$.

# Dieker-Yakir representation

Motivated by previous works of Siegmund and Yakir, Dieker & Yakir '14, [8] showed that

$$\mathcal{H} = \mathbb{E}\left\{ \frac{\sup_{t\in\mathbb{R}} e^{\widetilde{B}(t)}}{\int_{t\in\mathbb{R}} e^{\widetilde{B}(t)}} dt \right\} \tag{9}$$

**Q3'**: **What is the meaning of those representations?**

## Brown-Resnick max-stable rf's

With $E_k$'s iid unit exponential rv's define

$$X(t) = \max_{i \geq 1} \frac{Z_i(t)}{\sum_{k=1}^{i} E_k}, \quad t \in \mathcal{T} = \mathbb{R}^l \tag{10}$$

$Z_i$'s independent copies **of the representor** $Z$ given by

$$Z(t) = e^{\widetilde{B}(t)}, \quad t \in \mathcal{T}, \quad \widetilde{B}(t) = B(t) - Var(B(t))/2$$

**Stationarity** of $X$ is shown for:

- $B(t), t \in \mathbb{R}$ is a Brownian motion, Brown & Resnick '77, [9]
- $B(t) = tW, t \in \mathbb{R}$ with $W$ an $N(0,1)$ rv, Gale '80, [10]
- $B$ centered GRF's with stationary increments, De Haan et. al. '09, [11].

## General max-stable rf's

Consider a stochastically continuous max-stable rf $X(t), t \in \mathcal{T}$ with representator $Z(t), t \in \mathcal{T}$ satisfying for all compact $K \subset \mathcal{T}$

$$\mathbb{E}\left\{\sup_{t \in K} Z(t)\right\} < \infty, \quad \mathbb{P}\left\{\sup_{t \in \mathcal{T}} Z(t) > 0\right\} = 1 \tag{11}$$

Suppose that $X$ has unit Fréchet marginal's $e^{-1/x}, x > 0$, i.e.,

$$\mathbb{E}\{Z(t)\} = 1, \quad \forall t \in \mathcal{T}$$

If $x_i$'s are positive and $t_i$'s in $\mathcal{T}$

$$\mathbb{P}\{X(t_1) \le x_1, \ldots, X(t_k) \le x_k\} = e^{-\mathbb{E}\left\{\max_{1 \le i \le k} Z_i(t_i)/x_i\right\}} \tag{12}$$

# Tilt-shift formula

From (12), if

$$B^h Z(t) = Z(t-h), \quad t \in \mathcal{T}$$

is a representor of $X$ for all $h \in \mathcal{T}$, then $X$ is stationary.
As shown in H. '18, [12] stationarity of $X$ is equivalent with

$$\mathbb{E}\left\{Z(h)H(Z)\right\} = \mathbb{E}\left\{Z(0)H(B^h Z)\right\}, \quad \forall h \in \mathcal{T} \qquad (13)$$

for all $H : \mathrm{D}(\mathcal{T}, \mathbb{R}) \mapsto [0, \infty]$ measurable 0-homogeneous maps.

**Remark**: $Z$ is non-negative. We allow later for general $Z$.

# Tail measures

Given the **jointly measurable** rf $Z(t), t \in \mathcal{T}$ define the tail measure introduced in (Owada & Samorodnitsky '12)

$$\nu_Z[H] = \int_0^\infty \mathbb{E}\left\{H(r \cdot Z)\right\} r^{-2} dr$$

for all $H : \mathrm{D}(\mathcal{T}, \mathbb{R}) \mapsto [0, \infty]$ measurable, see also [13, 14].

Properties of $\nu_Z$:

- $\nu_Z$ is $-1$-homogeneous
- $\nu_Z$ is **shift-invariant** i.e.,

$$\nu_Z = \nu_{B^h Z}, \quad \forall h \in \mathcal{T}$$

$\iff$ $Z$ satisfies tilt-shift formula (13), details here [13, 15, 14].

# Regular variation of max-stable rf's

The max-stable rf $X$ with cadlag paths (Soulier '22, Bladt, H., Shevchenko '22, [15, 14]) **is regularly varying with tail measure** $\nu_Z$, i.e.,

$$\lim_{u\to\infty} \frac{\mathbb{E}\{H(X/u)\}}{\mathbb{P}\{X(0)>u\}} = \nu_Z[H], \quad u\to\infty \tag{14}$$

for all continuous bounded $H : \mathrm{D}(\mathcal{T},\mathbb{R}) \mapsto \mathbb{R}$ separated by the null map, i.e.,

$$\sup_{t\in K_H} |f(t)| < \varepsilon_H, \quad \forall f : H(f) = 0$$

for some compact $K_H \subset \mathcal{T}$ and $\varepsilon_H$ positive.

# Tail & spectral tail rf's

When the max-stable rf $X$ with cadlag paths **is stationary**, then we have (Soulier '22, Bladt, H., & Shevchenko '22)

$$\lim_{u\to\infty} \frac{\mathbb{E}\left\{H(X/u)\mathbb{I}(X(0)>u)\right\}}{\mathbb{P}\left\{X(0)>u\right\}} = \mathbb{E}\left\{H(Y)\right\}, \quad u\to\infty$$

for all continuous bounded $H : \mathrm{D}(\mathcal{T},\mathbb{R}) \mapsto \mathbb{R}$
separated by the null map.

- $Y$ is referred to as the **tail rf** of $X$
- $\Theta = Y/Y(0)$ is referred to as the spectral tail rf of $X$
- $Y = R\Theta$ with $R$ an 1-Pareto rv independent of $\Theta$

# Key relationships & questions

In more general settings, how to define and relate

- $$X, Z, \nu_Z, Y, \Theta$$
- max-stability
- shift-invariance
- regular variation for defining $Y$ **can be dropped**, see below
- Pickands & other constants?
- Applications?

# Definition of $\mathcal{H}_Z^{\mathcal{L}}$

Let $Z(t), t \in \mathcal{T}$ with $\mathcal{T} = \mathbb{R}^l$ be jointly measurable and separable. Suppose that for all compact $K \subset \mathcal{T}$ (11) holds, i.e.,

$$\mathbb{E}\left\{\sup_{t\in K}|Z(t)|\right\} < \infty, \quad \mathbb{P}\left\{\sup_{t\in\mathcal{T}}|Z(t)| > 0\right\} = 1$$

Given an additive subgroup $\mathcal{L}$ of $\mathcal{T}$ define **PiC's** by

$$\mathcal{H}_Z^{\mathcal{L}} = \lim_{T\to\infty} \mathcal{H}_Z^{\mathcal{L}}[T], \quad \mathcal{H}_Z^{\mathcal{L}}[T] = \frac{1}{T^l}\mathbb{E}\left\{\sup_{t\in[0,T]^l\cap\mathcal{L}}|Z(t)|\right\} \qquad (15)$$

**Of interest**: $\mathcal{L} = \mathcal{T}$ or $\mathcal{L}$ is a full rank lattice on $\mathbb{R}^l$.

# Relations with max-stable stationary $X$

**Lem 1**: If $Z$ satisfies the tilt-shift formula

$$\mathbb{E}\left\{|Z(h)|H(Z)\right\} = \mathbb{E}\left\{|Z(0)|H(B^h Z)\right\}, \quad \forall h \in \mathcal{T} \tag{16}$$

then the max-stable **stationary** $X$ with representor $|Z|$ has **extremal index** $\mathcal{H}_Z^{\mathcal{L}}$, i.e.,

$$\lim_{T\to\infty} \mathbb{P}\left\{ \sup_{t\in\delta\cap\mathbb{Z}^l\cap[0,T]^l} X(t) \le Tx \right\} = e^{-\mathcal{H}_Z^{\mathcal{L}}/x}, \quad x > 0$$

with

$$\mathcal{H}_Z^{\mathcal{L}} \in [0, \infty) \tag{17}$$

# 1-Homogeneous shift-invariant classes of rf's

Consider the class $\mathcal{K}[Z]$ of all jointly measurable rf's $Z(t), t \in \mathcal{T}$ defined on some complete $(\Omega, \mathcal{F}, \mathbb{P})$ such that (11) holds, i.e.,

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}} |\widetilde{Z}(t)| > 0\right\} = 1, \quad \mathbb{E}\left\{\sup_{t\in K} |\widetilde{Z}(t)|\right\} < \infty$$

for all compact $K \subset \mathcal{T}$ and $\widetilde{Z} \in \mathcal{K}[Z]$, see [16, 17].
Suppose the tilt-shift formula (13) is valid and further

$$\mathbb{E}\left\{|Z(h)|H(Z)\right\} = \mathbb{E}\left\{|\widetilde{Z}(h)|H(\widetilde{Z})\right\}, \quad \forall h \in \mathcal{T}, \widetilde{Z} \in \mathcal{K}[Z] \quad (18)$$

for all $H : \mathrm{D}(\mathcal{T}, \mathbb{R}) \mapsto [0, \infty]$ measurable 0-homogeneous maps.

# Stationary $Z$

When $Z$ is a stationary rf, the tilt-shift formula (16) is valid, so we can define $\mathcal{K}[Z]$.
Moreover, for such $Z$ we have

$$\mathcal{H}_Z^{\mathcal{L}} = 0$$

**Not any $Z$ defines a $\mathcal{K}[Z]$; non-stationary $Z$'s are of interest**.

# Refinements of tilt-shift formula

**Thm 1**: For all $\widetilde{Z} \in \mathcal{K}[Z]$ we have

$$\mathbb{E}\left\{H(Z)\right\} = \mathbb{E}\left\{H(B^h \widetilde{Z})\right\}, \quad \forall h \in \mathcal{T}$$

where $H : \mathrm{D}(\mathcal{T}, \mathbb{R}) \mapsto [0, \infty]$ is 1-homogeneous including

$$I_{\mathcal{T}} : f \mapsto \int_{\mathcal{T}} |f(t)| \lambda(dt)$$

Moreover, there $\exists$ a $\widetilde{Z} \in \mathcal{K}[Z]$ **stochastically continuous** satisfying

$$\mathbb{P}\left\{I_{\mathcal{T}}(\widetilde{Z}) > 0\right\} = 1 \tag{19}$$

# The spectral tail rf $\Theta$

Given a $\mathcal{K}[Z]$ determine $\Theta$ as the rf $Z/|Z(0)|$ under

$$\widehat{\mathbb{P}}(A) = \frac{1}{\mathbb{E}\left\{Z(0)\right\}}\mathbb{E}\left\{Z\mathbb{I}(A)\right\}, \qquad \forall A \in \mathscr{F}$$

# Properties of $\Theta$

- (13) is equivalent with: $\forall \Gamma \in \mathcal{H}_1$ including $I_{\mathcal{T}}$

$$\mathbb{E}\left\{|\Theta(h)|\Gamma(\Theta)\right\} = \mathbb{E}\left\{\mathbb{I}(|\Theta(-h)| \neq 0)\Gamma(B^h\Theta)\right\}, \quad \forall h \in \mathcal{T}$$

- $\mathbb{P}\left\{|\Theta(0)| = 1\right\} = 1$
- A third property follows from $\mathbb{E}\left\{\sup_{t\in K}|Z(t)|\right\} < \infty$

Given $\Theta$ satisfying the above properties, a shift-invariant $\mathcal{K}[Z]$ can be constructed [18, 19, 16].

# The tail rf $Y$

We can define the tail rf $Y$ by

$$Y = R\Theta$$

with $R$ a 1-Pareto rv independent of $\Theta$.
We can choose $\Theta$ to be stochastically continuous.
This implies

- $\mathbb{P}\{I_{\mathcal{T}}(\Theta) > 0\} = 1$
- $Y$ is stochastically continuous
- $S_{\mathcal{T}}(Y) = \int_{t \in \mathcal{T}} \mathbb{I}(|Y(t)| > 1)\lambda(dt) > 0$ almost surely

# Properties of $Y$

- $\forall$ **measurable** $\Gamma$ including $I_{\mathcal{T}}, S_{\mathcal{T}}$, see [13, 14, 16, 20]

$$\mathbb{E}\left\{\Gamma(xB^{h}Y)\right\} = x\mathbb{E}\left\{\Gamma(Y)\mathbb{I}(|xB^{-h}Y/x| > 1)\right\}, \forall h \in \mathcal{T}, x > 0$$

- $|Y(0)|$ is an 1-Pareto rv
- For all compact $K \subset \mathcal{T}$ with positive Lebesgue measure

$$\int_{t \in K} \mathbb{E}\left\{\frac{1}{\int_{s \in K} \mathbb{I}(|Y(s-t)| > 1)\lambda(ds)}\right\}\lambda(dt) < \infty \qquad (20)$$

Conversely, given $Y$ satisfying the above properties,
a shift-invariant $\mathcal{K}[Z]$
can be constructed (Kulik & Soulier '20, Soulier '22, H. '24).

# Cluster RF's $Q$

Let $Q(t), t \in \mathcal{T}$ be jointly measurable and separable.
Suppose that for all compact $K \subset \mathcal{T}$

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}}|Q(t)| > 0\right\} = 1, \quad \int_{\mathcal{T}} \mathbb{E}\left\{\sup_{t\in K}|Q(v-t)|\right\}\lambda(dv) < \infty \quad (21)$$

If $N$ is independent of $Q$ with density $p(t) > 0, t \in \mathcal{T}$, then

$$Z(t) = \frac{B^N Q(t)}{p(N)}, \quad t \in \mathcal{T}$$

defines a shift-invariant $\mathcal{K}[Z]$.

**Q4**: Given $\mathcal{K}[Z]$ does a corresponding $Q$ exist?
If yes, how to construct $Q$?

# Example: Brown-Resnick $\mathcal{K}[Z]$

Consider for $W$ a **centered GRF** with stationary increments

$$Z(t) = e^{W(t) - Var(W(t))/2}, \quad t \in \mathcal{T}$$

$Z$ defines a shift-invariant $\mathcal{K}[Z]$ and a shift-invariant $\nu_Z$.
For this case

$$\Theta(t) = e^{W(t) - W(0) - \gamma(t)/2}, \quad t \in \mathcal{T}$$

with variogram $\gamma(t) = Var(W(t) - W(0))$, and

$$Y(t) = e^{E + W(t) - W(0) - \gamma(t)/2}, \quad t \in \mathcal{T}$$

with $E$ a unit exponential rv independent of $W$.

# Existence of $Q$

**Lem 2:** Given a shift-invariant $\mathcal{K}[Z]$, then a stochastically continuous cluster rf $Q$ exists **iff** almost surely

- $I_{\mathcal{T}}(Z) = \int_{\mathcal{T}} |Z(t)| \lambda(dt) < \infty$
- $I_{\mathcal{T}}(\Theta) < \infty$
- $S_{\mathcal{T}}(Y) = \int_{\mathcal{T}} \mathbb{I}(|Y(t)| > 1) \lambda(dt) < \infty$

or one of the above holds with $\mathcal{T}$ **substituted** $\mathbb{Z}^l$ and $\lambda$ **substituted** by the counting measure on $\mathbb{Z}^l$.

# Constructions of different $Q$'s

**Thm 2**: If

$$\mathbb{P}\left\{ \int_{\mathcal{T}} |Z(t)| \lambda(dt) < \infty \right\} = 1$$

then stochastically continuous $Q$ can be constructed.
We have

$$Q = c\Theta, \quad c^{-1} = I_{\mathcal{T}}(\Theta)$$

$$Q = cY, \quad c^{-1} = \sup_{t \in \mathcal{T}} |Y(t)| S_{\mathcal{T}}(Y)$$

**Remark**: Other constructions possible by employing anchoring maps, [18, 15, 17].

# Generalised Picaknds constants

**Lem 3**: If $\mathcal{L} = \{Ax, x \in \mathbb{Z}^l\}$, where $A$ is a $l \times l$ real, non-singular matrix or $\mathcal{L} = \mathcal{T}$ and $\mathcal{K}[Z]$ possesses a cluster rf $Q$, then

$$\mathcal{H}_Z^{\mathcal{L}} = \frac{1}{\Delta(\mathcal{L})} \mathbb{E}\left\{ \sup_{t \in \mathcal{L}} |Q(t)| \right\}$$

where $\Delta(\mathcal{L})$ is the volume of $\{Ax, x \in [0,1)^l\}$.

**Remark**: **a)** New representations for B-R $X$ follow.
**b)** When $\mathcal{L} = \mathcal{T}$, then set $\Delta(\mathcal{L}) = 1$.
**c)** More results in [18, 17].

# Rosiński representations (RR's)

If for $\mathcal{K}[Z]$ exists a cluster rf $Q$ and $Z$ is non-negative, then for the corresponding max-stable $X$ we have a new representation for its fidi's.
Namely, for $x_i$'s positive and $t_i$'s in $\mathcal{T}$

$$\mathbb{P}\left\{X(t_i) \le x_i, 1 \le i \le k\right\} = e^{-\mathbb{E}\left\{\int_{\mathcal{T}} \max_{1 \le i \le k} Q_i(t_i - s)/x_i \lambda(ds)\right\}} \quad (22)$$

**Remark**: **a)** **RR**'s also called M3 or moving maxima representation.
**b)** New $Q$'s lead to new **RR**'s

# Shift-representations of tail measures

If $\nu_Z$ is a shift-invariant tail measure and $Z$ has a cluster rf $Q$, then $\nu_Z$ can be defined as a mixture of $\nu_{B^h Q}$.
Specifically, we have

$$\nu_Z[H] = \int_{\mathcal{T}} \nu_{B^h Q}[H] \lambda(dh)$$

with $\lambda$ the Lebesgue measure.
Some properties of such tail measures can be explored in terms of $Y$ or $\Theta$ and $Q$, [18, 15, 17].

# New self-similar covariance functions

Tail measures $\nu_Z$ appear in the limit of different functionals (Kulik & Soulier '20). Given a cluster rf $Q$ with corresponding tail rf $Y$, under weak assumptions

$$K(s,t) = s \int_{\mathcal{T}} \mathbb{P}\{|Y(h)| > t/s\}\,\lambda(dh), \quad 0 < s \le t \tag{23}$$

defines a covariance kernel.

Extensions and further ideas is work in progress, H. 25+.

**Many thanks for your attention and interest!**

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