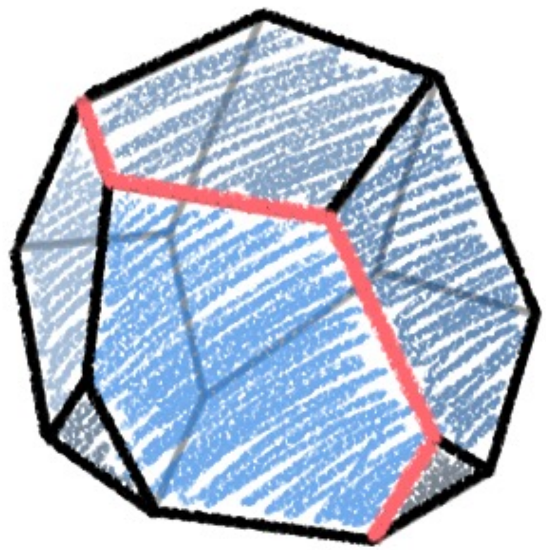
Computational, combinatorial, and geometric aspects of linear optimization



Antoine Deza, McMaster

based on joint works with:

Shmuel Onn, Technion,

Sebastian Pokutta, ZIB / TU Berlin

Lionel Pournin, Paris XIII

Linear optimization

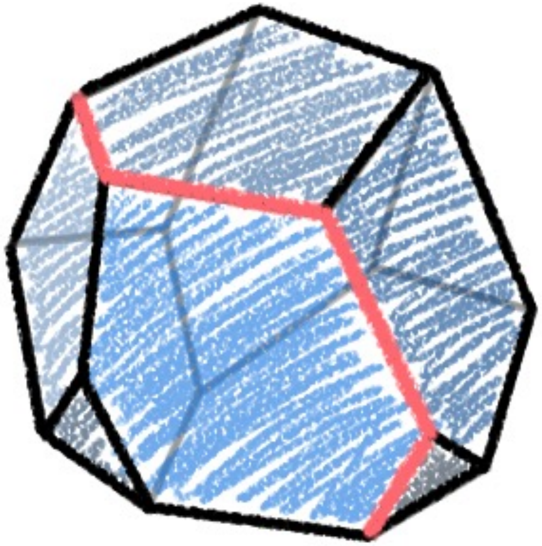
Given an *n*-dimensional vector ***b*** and an *n* x *d* matrix ***A***
find, in any, a *d*-dimensional vector ***x*** such that :

Ax* = *b ***Ax* = *b***, ***x* ≥ 0**

linear algebra **linear optimization**

"Can linear optimization be solved in ***strongly polynomial*** time?"
is listed by Smale as one of the top problems for the XXI century

Strongly polynomial *:* algorithm ***independent*** from
the ***input data length*** and polynomial in *n* and *d.*

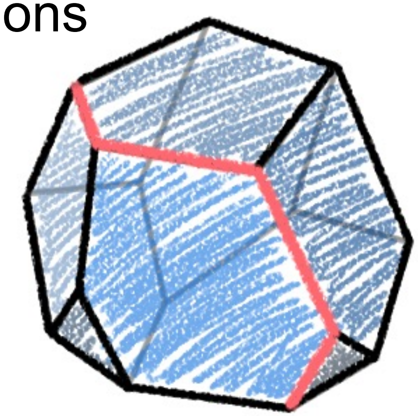


Linear optimization algorithms
simplex methods

Given an ***n***-dimensional vector ***b*** and an ***n*** x ***d*** (full row-rank) matrix ***A*** and a ***d***-dimensional cost vector ***c***, solve : max $\{ \boldsymbol{c}^{T}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \}$

Simplex methods (Dantzig 1947): pivot-based, combinatorial, ***not proven to be polynomial***, efficient in practice

- start from a ***feasible basis***
- use a ***pivot rule***
- find an optimal solution after a ***finite number*** of iterations
- most known pivot rules are known to be ***exponential*** (worst case); ***efficient*** implementations exist



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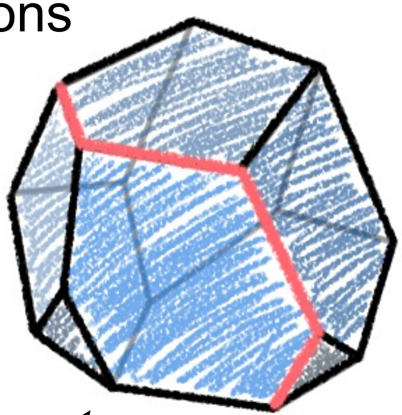

- $\delta(d,n)$ largest diameter over all (d,n)-polytopes: lower bound on the number of simplex pivots required in the ***worst case.***

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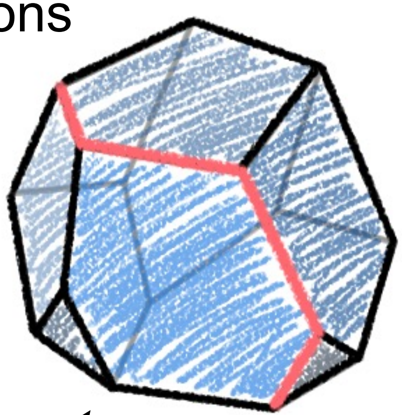

- δ(***d***,***n***) largest diameter over all (***d***,***n***)-polytopes: lower bound on the number of simplex pivots required in the ***worst case.***
- δ(**4**,**12**)= δ(**5**,**12**) =7 [Bremner-Deza-Hua-Schewe 2013] , …

Linear optimization algorithms
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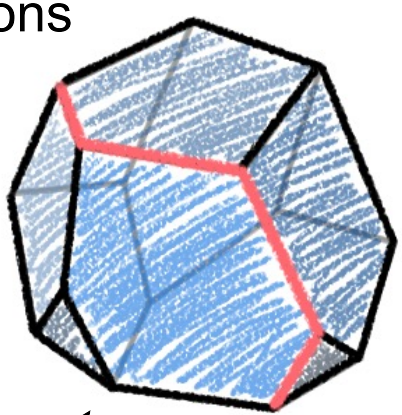

- $\delta(d,n)$ largest diameter over all (d,n)-polytopes: lower bound on the number of simplex pivots required in the ***worst case.***
- *random spherical* polytopes [Bonnet, Dadush, Grupel, Huiberts, Livshyts 2021], smooth analysis [Huiberts, Lee, Zhang 2024], …

Linear optimization algorithms
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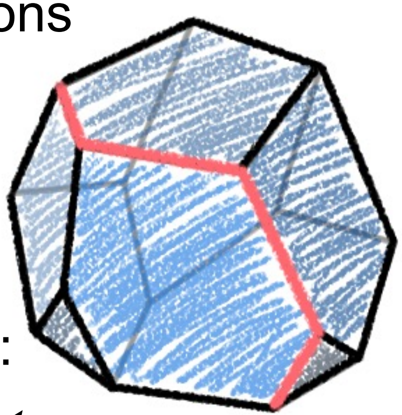

- $\delta(d,k)$ largest diameter over all **lattice** (d,k)-polytopes:

lower bound on the number of simplex pivots in the ***worst case.***

Linear optimization algorithms
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Given an n-dimensional vector $\boldsymbol{b}$ and an n x d (full row-rank) matrix $\boldsymbol{A}$ and a d-dimensional cost vector $\boldsymbol{c}$, solve : $\max \{ \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \}$

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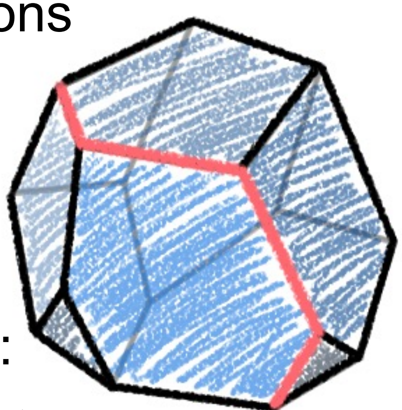


- $\delta(d,k)$ largest diameter over all **lattice** (d,k)-polytopes:

lower bound on the number of simplex pivots in the ***worst case.***

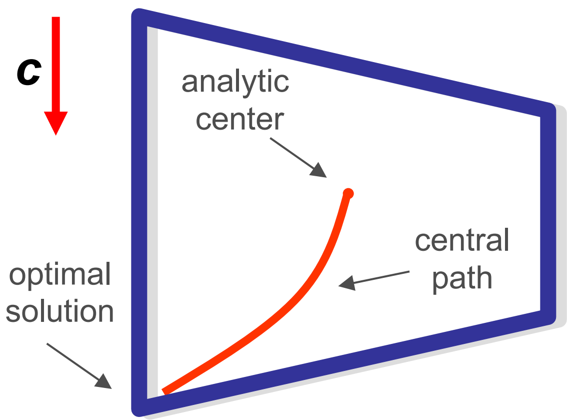
- [Del Pia-Michini 2022] preprocessing and scaling algorithm yielding simplex paths that are short relative to $\delta(d,k)$

Linear optimization algorithms
(central path following) interior point methods

Given an n-dimensional vector $\boldsymbol{b}$ and an n x d (full row-rank) matrix $\boldsymbol{A}$ and a d-dimensional cost vector $\boldsymbol{c}$, solve : max $\{ \boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \}$

Interior Point Methods :
path-following, ***polynomial***, efficient in practice

- start from the ***analytic center***
- follow the ***central path***
- converge to an optimal solution in $O(\sqrt{n}L)$ iterations
  (L: input data length)



$$\max \quad c^T x - \mu \sum_i \ln(b - Ax)_i$$

μ : central path parameter
$x \in P : Ax \leq b$

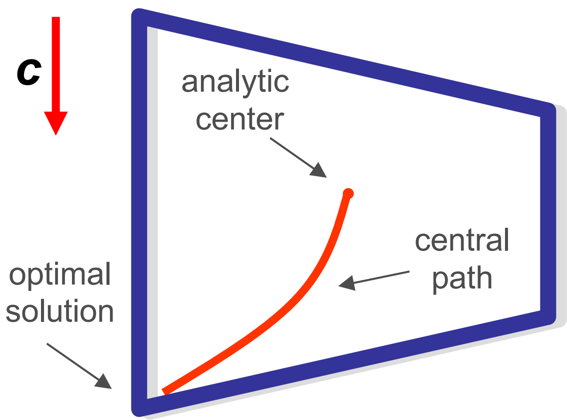
Linear optimization diameter and curvature

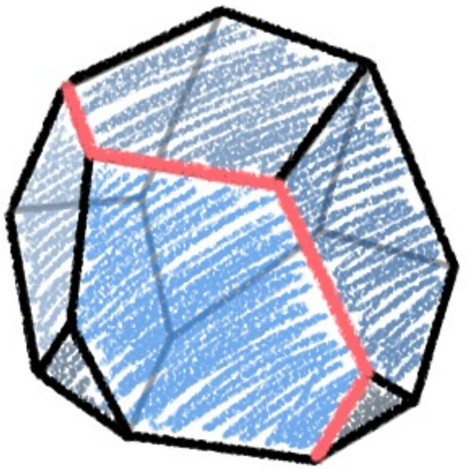
Diameter (*of a polytope*) :

lower bound for the number of iterations for *pivoting* **simplex methods**

Curvature (*of the central path associated to a polytope*) :

large curvature indicates large number of iterations for *path following* **interior point methods**





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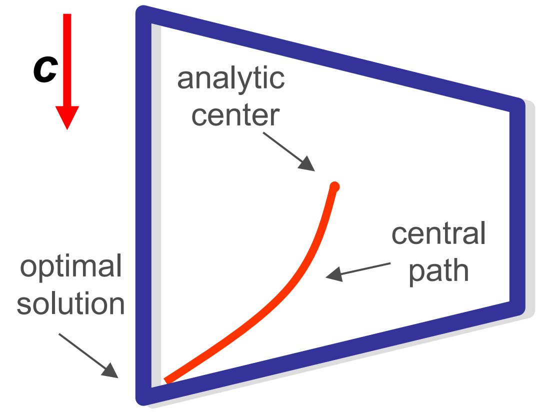
- [Dedieu-Malajovich-Shub 2005]
  Average curvature of the central path is bounded by $2\pi d$
  (analogue of Haimovich (Borgwardt 1987) analysis of the simplex)

- ***Conjecture*** [Dedieu-Malajovich 2005]
  Curvature of the central path is $O(d)$



- [Deza-Terlaky-Zinchenko 2008]
  Counterexample to Dedieu-Malajovich conjecture

- ***Conjecture*** [Deza-Terlaky-Zinchenko 2008]
  Curvature of the central path is $O(n,d)$
  (continuous analogue of the Hirsch conjecture)

Linear optimization

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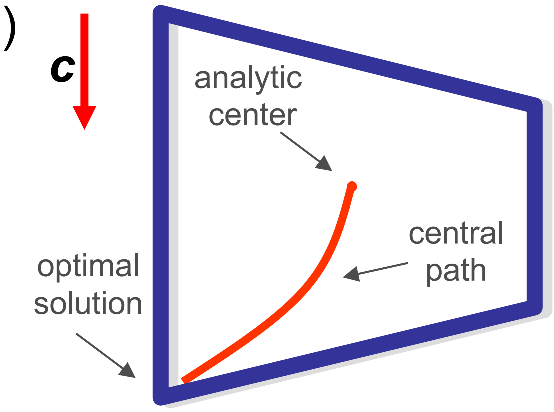
- [Allamigeon, Benchimol, Gaubert, Joswig 2018]
(logarithmic barrier) ***Interior point methods***
are ***not strongly polynomial***
(tropical counterexample to continuous Hirsch conjecture)

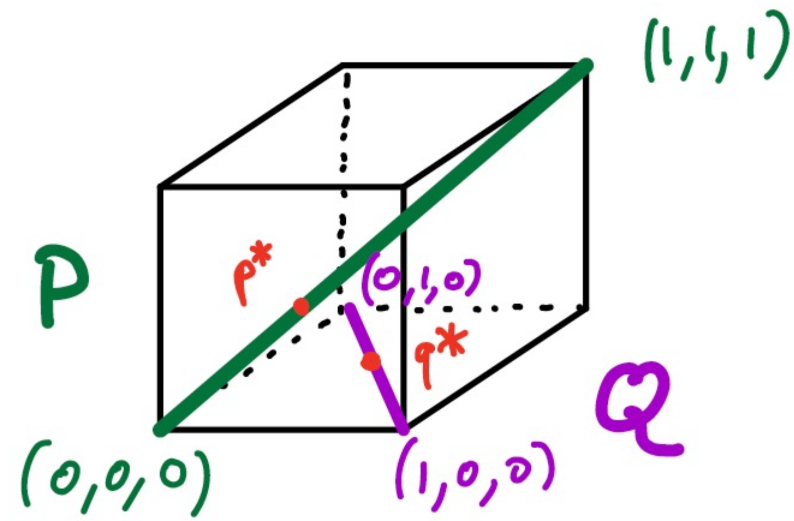


- [Allamigeon, Gaubert, Vandame 2022]
(self-concordant barrier) ***Interior point methods***
are ***not strongly polynomial***

- Further results: [Allamigeon-Dadush-Loho-Natura-Végh 2024] , ….

Kissing Polytopes

Q. How close can two disjoint d-dimensional $0/1$-polytopes be?



Motivation: this minimal distance appears in complexity bounds of algorithms such as **von Neumann's alternating projections**

Complexity of certifying that $P \cap Q = \emptyset$ is

$$O\left(\frac{1}{d(P,Q)^2}\right)$$

Alternating Linear Minimization: Revisiting von Neumann's alternating projections [Braun-Pokutta-Weismantel 2022]

Main question, motivation, related works

Related works: $\mathcal{V}$: vertex set of a polytope P

- **Facial distance** of P is:

$$\Phi(P) = \min\Big\{d\Big(F, \mathbf{conv}(\mathcal{V} \setminus F)\Big) : F \text{ proper } \textbf{face} \text{ of } P\Big\}$$

[Peña-Rodriguez 2018], [Gutman-Peña 2018], [Peña 2019]

- **Vertex-facet distance** of P is:

$$\Delta(P) = \min\Big\{d\Big(\mathbf{aff}(F), \mathrm{conv}(\mathcal{V} \setminus F)\Big) : F \textbf{ facet} \text{ of } P\Big\}$$

[Beck-Shtern 2017]

- **Minimal** vertex-facet distance of all d-dimensional **0/1-simplex** S

$$\frac{1}{\sqrt{2}^{d \log d - 2d + o(d)}} \leq \min \Delta(\mathrm{S}) \leq \frac{1}{\sqrt{2}^{d \log d - 4d + o(d)}}$$

[Alon-Vũ 1997]

Main results

- If P and Q are **disjoint** d-dimensional **0/1**-polytopes, then

$$\frac{1}{\sqrt{d}^{3d+2}} \leq d(P,Q)$$

- For any large enough d, there exist two **disjoint** d-dimensional **0/1**-polytopes P and Q such that

$$d(P,Q) \leq \frac{1}{\sqrt{d}^{\sqrt{d}}}$$

Main results

- **Theorem**: For any large enough d, there exist two **disjoint** d-dimensional **0/1**-polytopes P and Q such that

$$\frac{1}{\sqrt{d}^{3d+2}} \leq d(P, Q) \leq \frac{1}{\sqrt{d}^{\sqrt{d}}}$$

$\Rightarrow$ Similar bounds for **lattice** (d, k)**-polytope** (that is, convex hull of points drawn from $\{0, 1, \ldots, k\}^d$)

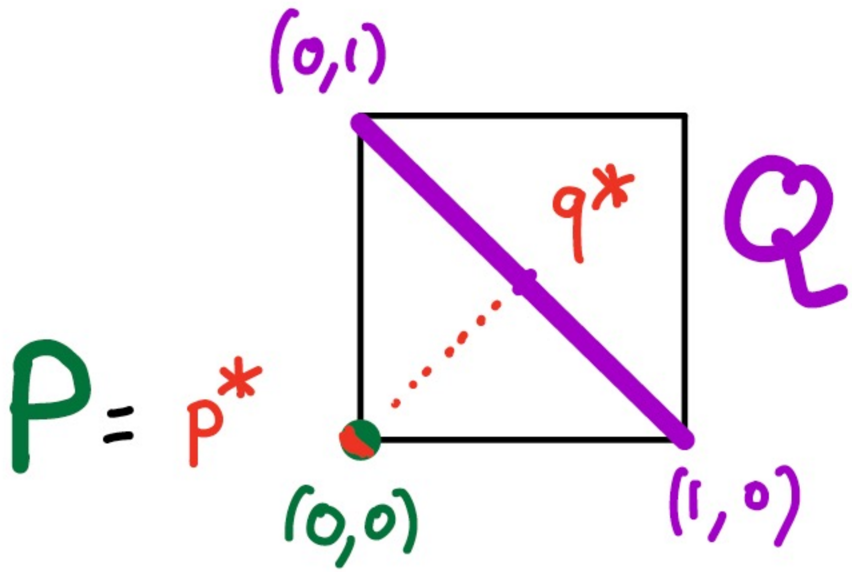
$\Rightarrow$ New bounds for the **minimal facial distance** $\Phi(P)$ over of all **lattice** (d, k)**-polytopes** P

$\Rightarrow$ Similar bounds for **rational polytopes** in terms of **binary encoding length** and dimension

How close can disjoint 0/1-polytopes be?

$\Rightarrow \quad \varepsilon(d,1) = \min\{d(P,Q) : P, Q$ disjoint d-dimensional 0/1-polytopes$\}$

What about dimension 2 ?



$$d(P,Q) = 0, \frac{1}{\sqrt{2}}, 1 \text{ or } \sqrt{2} \qquad \Rightarrow \qquad \varepsilon(2,1) = \frac{1}{\sqrt{2}}$$

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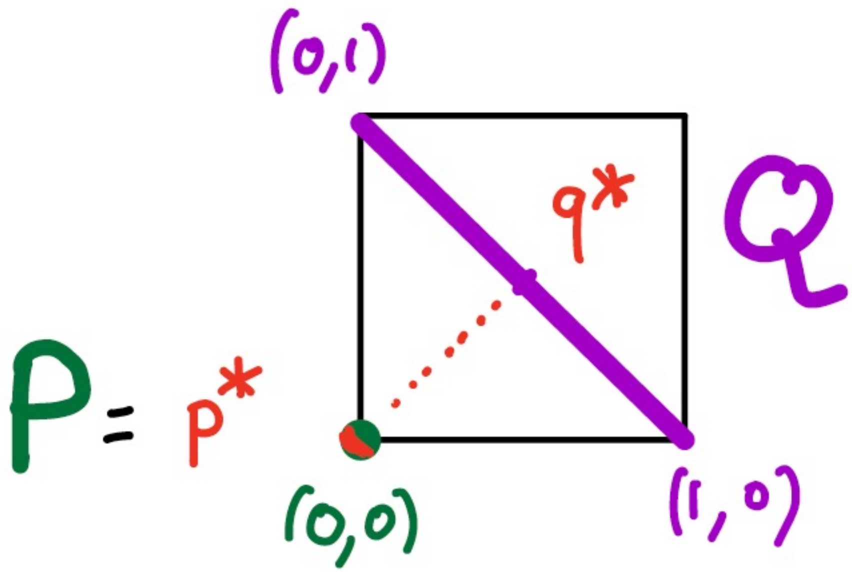

$$d(P,Q) = 0, \frac{1}{\sqrt{2}}, 1 \text{ or } \sqrt{2} \qquad \Rightarrow \qquad \varepsilon(2,1) = \frac{1}{\sqrt{2}}$$

- $\varepsilon(2,1)$ achieved as $d(p^*,q^*)$ with q^* **not** a vertex of Q
- If P or Q is reduced to a **vertex**, $d(P,Q) \geq \frac{1}{\sqrt{d}}$ for **any** d

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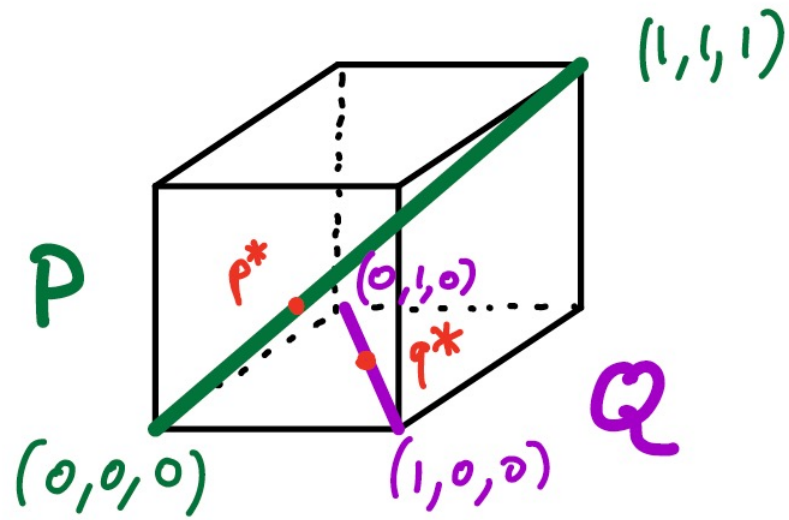
What about dimension 3 ?



$$d(P,Q) = 0, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{2}{\sqrt{6}}, 1, \frac{3}{\sqrt{6}}, \sqrt{2} \text{ or } \sqrt{3} \quad \Rightarrow \quad \varepsilon(3,1) = \frac{1}{\sqrt{6}}$$

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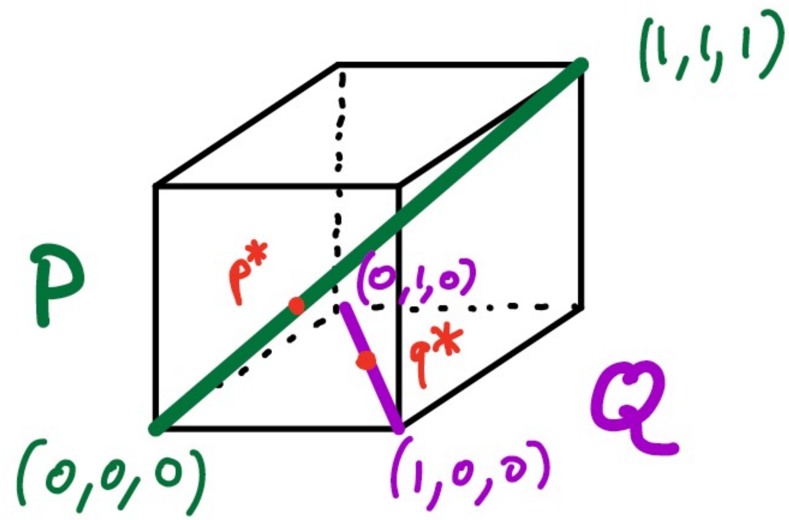

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- $\varepsilon(3,1)$ achieved as $d(p^*,q^*)$ with both p^* and q^* **not** vertices of P,Q
- If both P and Q are **segments**, $d(P,Q) \geq \frac{1}{\sqrt{6}}$ for **any** d

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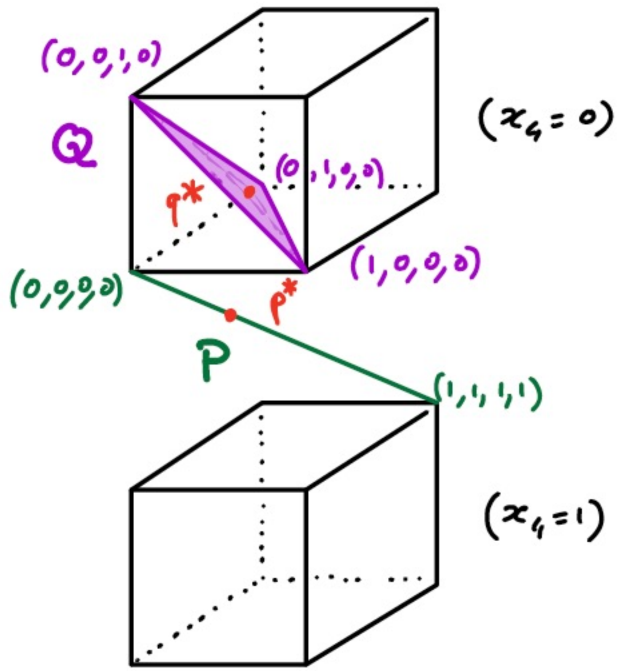
What about dimension d?



P: segment $[(0,0,\ldots,0),(1,1,\ldots,1)]$

Q: $(d-2)$-simplex with vertices $(1,0,..0,\mathbf{0}),(0,1,..0,\mathbf{0}),\ldots,(0,0,..1,\mathbf{0})$

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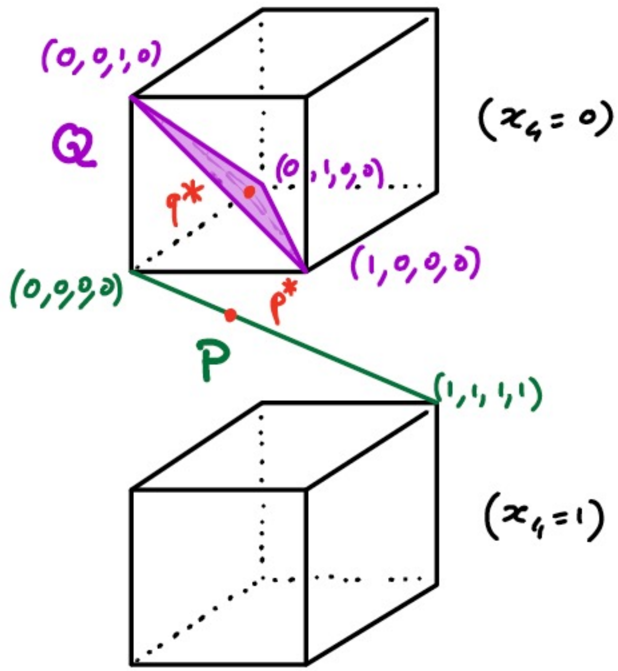

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Q: $(d-2)$-simplex with vertices $(1,0,..0,\mathbf{0}),(0,1,..0,\mathbf{0}),\ldots,(0,0,..1,\mathbf{0})$

$p^* = \frac{1}{d}(1,1,\ldots,1)$, $q^* = \frac{1}{d-1}(1,1,\ldots,1,0)$, $d(p^*,q^*) = \frac{1}{\sqrt{d(d-1)}}$

How close can disjoint 0/1-polytopes be?

$\Rightarrow\ \varepsilon(d, 1) = \min\{d(P, Q) : P, Q$ disjoint d-dimensional 0/1-polytopes$\}$

- **Obtaining** a non-trivial **upper bound** on $\varepsilon(d, 1)$ requires to exhibit two **disjoint polytopes** that are **very close**

- **Obtaining** a **lower bound** on $\varepsilon(d, 1)$ requires investigating the **geometric setup**

Geometric setup for rational polytopes

Remark: If P and Q are **rational polytopes** then $d(P, Q)^2$ is **rational**

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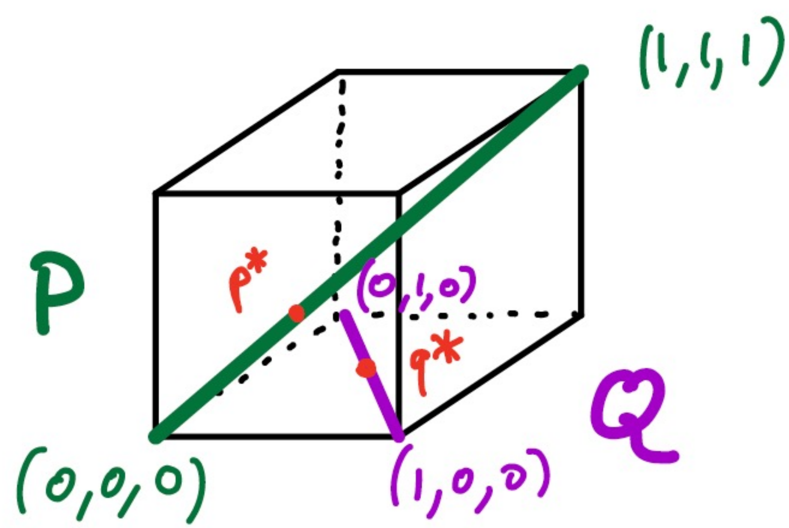

$d(P,Q)$ achieved as $d(p^*,q^*)$ with $(p^* - q^*)$ **orthogonal** to a **unique** face of P (resp. Q) containing p^* (resp. q^*) in its **relative interior**

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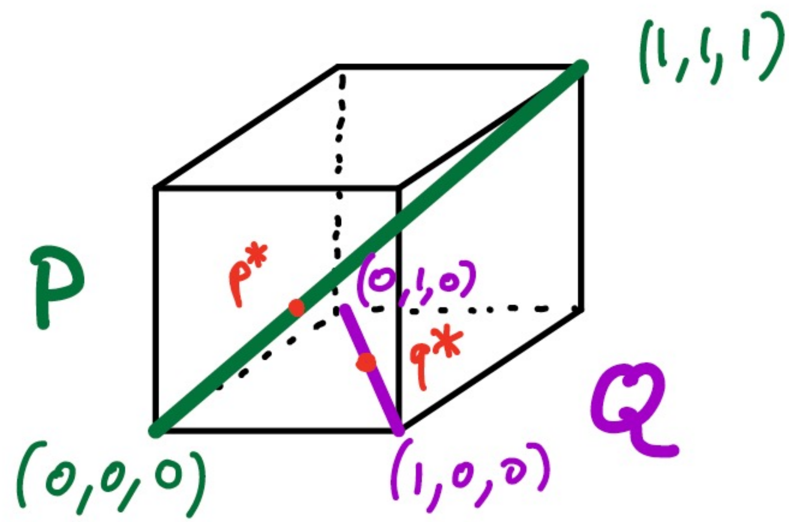

$d(P,Q)$ achieved as $d(p^*,q^*)$ with $(p^* - q^*)$ **orthogonal** to a **unique** face of P (resp. Q) containing p^* (resp. q^*) in its **relative interior**

- (essentially) **Cramer**'s rule yields $(p^* - q^*)$ is **rational**

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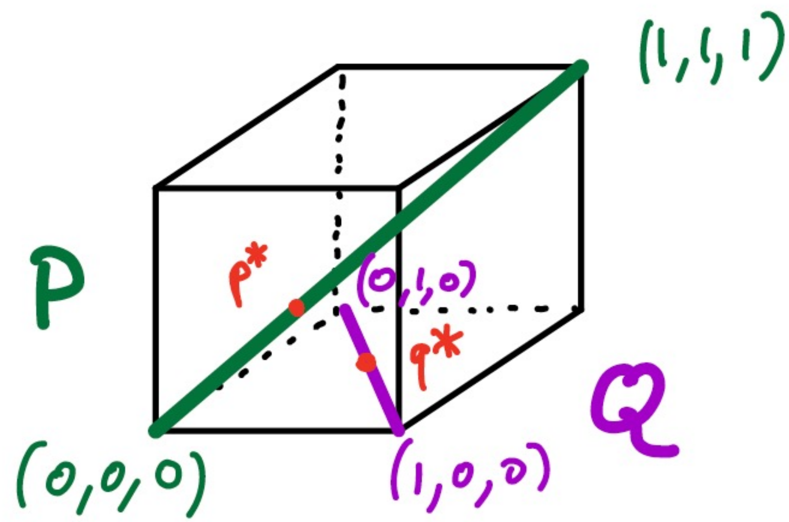

$d(P,Q)$ achieved as $d(p^*,q^*)$ with $(p^* - q^*)$ **orthogonal** to a **unique** face of P (resp. Q) containing p^* (resp. q^*) in its **relative interior**

- (essentially) **Cramer**'s rule yields $(p^* - q^*)$ is **rational**
- (in addition) $d(P,Q)$ can be achieved with **both** p^* and q^* **rational**

Geometric setup for rational polytopes

Remark: If P and Q are **rational polytopes** then $d(P,Q)^2$ is **rational**

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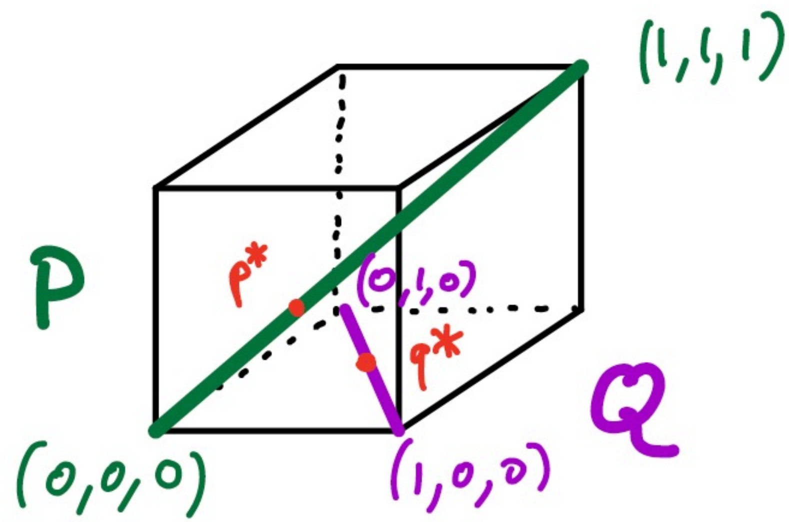
Direct geometric proof yields a lower bound:



$d(P,Q)$ achieved as $d(p^*,q^*)$ with (p^*-q^*) **orthogonal** to a **unique** face of P (resp. Q) containing p^* (resp. q^*) in its **relative interior**

- (essentially) **Cramer**'s rule yields (p^*-q^*) is **rational**
- (essentially) **Hadamard**'s inequality yields $d(p^*,q^*) \geq \frac{1}{\sqrt{d}^{3d+2}}$.

How close can disjoint 0/1-polytopes be?

$\varepsilon(d, 1) = \min\{d(P, Q) : P, Q$ disjoint d-dimensional 0/1-polytopes$\}$

$$\frac{1}{\sqrt{d}^{3d+2}} \leq \varepsilon(d, 1) \leq \quad ?$$

- **Obtaining** a non-trivial **upper bound** on $\varepsilon(d, 1)$ requires to exhibit two **disjoint polytopes** that are **very close**

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- **Obtaining** a non-trivial **upper bound** on $\varepsilon(d, 1)$ requires to exhibit two **disjoint polytopes** that are **very close**

∘ **Intuition**: $d(P, Q)$ might be not too different from $d(\text{aff}(P), \text{aff}(Q))$

Example: For $i \leq d - 1$, consider two consecutive hyper-simplices:

P = convex hull of all $x \in \{0, 1\}^d$ such that $x_1 + x_2 + \cdots + x_d = i$
Q = convex hull of all $x \in \{0, 1\}^d$ such that $x_1 + x_2 + \cdots + x_d = i + 1$

$\mathbf{a} = (1, 1, \ldots, 1)$ **orthogonal** to both aff(P) and aff(Q)

$$\Rightarrow \quad d(P, Q) = d(\text{aff}(P), \text{aff}(Q)) = \frac{1}{\sqrt{d}} = \frac{1}{||\mathbf{a}||}$$

How close can disjoint 0/1-polytopes be?

$\varepsilon(d,1) = \min\{d(P,Q) : P,Q$ disjoint d-dimensional 0/1-polytopes$\}$

$$\frac{1}{\sqrt{d}^{3d+2}} \leq \varepsilon(d,1) \leq \quad ?$$

- **Obtaining** a non-trivial **upper bound** on $\varepsilon(d,1)$ requires to exhibit two **disjoint polytopes** that are **very close**

∘ **Intuition**: $d(P,Q)$ might be not too different from $d(\text{aff}(P),\text{aff}(Q))$

Example: For $i \leq d-1$, consider two consecutive hyper-simplices:

P = convex hull of all $x \in \{0,1\}^d$ such that $x_1 + x_2 + \cdots + x_d = i$
Q = convex hull of all $x \in \{0,1\}^d$ such that $x_1 + x_2 + \cdots + x_d = i+1$

$\mathbf{a} = (1,1,\ldots,1)$ **orthogonal** to both aff(P) and aff(Q)

$$\Rightarrow \quad d(P,Q) = d(\text{aff}(P),\text{aff}(Q)) = \frac{1}{\sqrt{d}} = \frac{1}{||\mathbf{a}||}$$

∘ Can we make $||\mathbf{a}||$ **larger** while **maintaining** $d(P,Q) = d(\text{aff}(P),\text{aff}(Q))$?

How close can disjoint 0/1-polytopes be?

$\varepsilon(d, 1) = \min\{d(P, Q) : P, Q$ disjoint d-dimensional 0/1-polytopes$\}$

Consider **much larger** $||\mathbf{a}||$

- $\mathbf{a} \approx (\underbrace{1, \ldots, 1}_{\sqrt{d} \text{ times}}, \underbrace{-\sqrt{d}, \ldots, -\sqrt{d}}_{\sqrt{d} \text{ times}}, \ldots, \underbrace{(-\sqrt{d})^{\sqrt{d}}, \ldots, (-\sqrt{d})^{\sqrt{d}}}_{\sqrt{d} \text{ times}})$
- $P =$ convex hull of all $x \in \{0,1\}^d$ such that $\mathbf{a} \cdot x = \mathbf{0}$
- $Q =$ convex hull of all $x \in \{0,1\}^d$ such that $\mathbf{a} \cdot x = \mathbf{1}$

Maintaining $d(P, Q) = d(\text{aff}(P), \text{aff}(Q))$:
(careful) **convex combination** of (carefully) chosen vertices in P (resp. Q) yields a point $p^* \in P$ (resp. $q^* \in Q$) such that, for d large enough

$$d(p^*, q^*) \approx \frac{1}{||\mathbf{a}||} \approx \frac{1}{\sqrt{d}^{\sqrt{d}}}$$

How close can disjoint lattice polytopes be?

$\varepsilon(d, 1) = \min\{d(P, Q) : P, Q$ disjoint d-dimensional $0/1$-polytopes$\}$

$$\frac{1}{\sqrt{d}^{3d+2}} \leq \varepsilon(d, 1) \leq \frac{1}{\sqrt{d}^{\sqrt{d}}}$$

$\varepsilon(d, k) = \min\{d(P, Q) : P, Q$ disjoint lattice (d, k)-polytopes$\}$

$$\frac{1}{k^{2d-1}\sqrt{d}^{3d+2}} \leq \varepsilon(d, k) \leq \frac{1}{(k\sqrt{d})^{\sqrt{d}}}$$

How close can disjoint lattice polytopes be?

$$\varepsilon(d,1) = \min\{d(P,Q) : P,Q \text{ disjoint } d\text{-dimensional } 0/1\text{-polytopes}\}$$

$$\frac{1}{\sqrt{d}^{3d+2}} \le \varepsilon(d,1) \le \frac{1}{\sqrt{d}^{\sqrt{d}}}$$

$$\varepsilon(d,k) = \min\{d(P,Q) : P,Q \text{ disjoint lattice } (d,k)\text{-polytopes}\}$$

$$\frac{1}{k^{2d-1}\sqrt{d}^{3d+2}} \le \varepsilon(d,k) \le \frac{1}{(k\sqrt{d})^{\sqrt{d}}}$$

Q. How to **compute** the minimal distance between disjoint lattice polytopes for small dimension d and small range k?

How close can disjoint lattice polytopes be?

$$\varepsilon(d,k) = \min\{d(P,Q) : P, Q \text{ disjoint lattice } (d,k)\text{-polytopes}\}$$

Exploiting the **symmetries** of the cube, proving that one can assume that P is a simplex such that $1 \leq \dim(P) \leq \lfloor \frac{d}{2} \rfloor$, and Q is a simplex such that $\dim(Q) = d+1-\dim(P)$, significantly reduce the huge search space and allow for the **computation** of $\varepsilon(d,k)$ for small (d,k):

(d,k)	1	2	3	4	5	6
$\frac{1}{\varepsilon(2,k)}$	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{13}$	$\sqrt{25}$	$\sqrt{41}$	$\sqrt{61}$
$\frac{1}{\varepsilon(3,k)}$	$\sqrt{6}$	$\sqrt{50}$	$\sqrt{299}$			
$\frac{1}{\varepsilon(4,k)}$	$\sqrt{18}$					
$\frac{1}{\varepsilon(5,k)}$	$\sqrt{58}$					

How close can disjoint lattice polytopes be?

$$\varepsilon(d,k) = \min\{d(P,Q) : P, Q \text{ disjoint lattice } (d,k)\text{-polytopes}\}$$

[Deza-Liu-Pournin 2024]

- $\varepsilon(2,k) = \frac{1}{\sqrt{(k-1)^2+k^2}}$ for $k \geq 2$
- $\varepsilon(3,k) = \frac{1}{\sqrt{2(2k^2-4k+5)(2k^2-2k+1)}}$ for $k \geq 4$

Determination of $\varepsilon(3,k)$ amounts to minimize

$$\frac{|\mathbf{f}(x)|}{\sqrt{\mathbf{g}(x)}}$$

$$\mathbf{f}(x) = x_1(x_6x_8 - x_5x_9) + x_2(x_4x_9 - x_6x_7) + x_3(x_5x_7 - x_4x_8)$$

$$\mathbf{g}(x) = (x_1x_5 - x_2x_4)^2 + (x_1x_6 - x_3x_4)^2 + (x_2x_6 - x_3x_5)^2$$

such that $\mathbf{f}(x) \neq 0$, $\mathbf{g}(x) \neq 0$, $-k \leq x_i \leq k$, x_i integer for $i = 1, 2, \ldots, 9$

How close can disjoint lattice polytopes be?

$$\varepsilon(d,k) = \min\{d(P,Q) : P, Q \text{ disjoint lattice } (d,k)\text{-polytopes}\}$$

(d,k)	1	2	3	4	...	k
$\frac{1}{\varepsilon(2,k)}$	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{13}$	$\sqrt{25}$	...	$\sqrt{(k-1)^2+k^2}$
$\frac{1}{\varepsilon(3,k)}$	$\sqrt{6}$	$\sqrt{50}$	$\sqrt{299}$	$\sqrt{1050}$	...	$\sqrt{2(2k^2-4k+5)(2k^2-2k+1)}$
$\frac{1}{\varepsilon(4,k)}$	$\sqrt{18}$					
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How close can disjoint lattice polytopes be?

$$\varepsilon(d,k) = \min\{d(P,Q) : P, Q \text{ disjoint lattice } (d,k)\text{-polytopes}\}$$

(d,k)	1	2	3	4	...	k
$\frac{1}{\varepsilon(2,k)}$	$\sqrt{2}$	$\sqrt{5}$	$\sqrt{13}$	$\sqrt{25}$	...	$\sqrt{(k-1)^2+k^2}$
$\frac{1}{\varepsilon(3,k)}$	$\sqrt{6}$	$\sqrt{50}$	$\sqrt{299}$	$\sqrt{1050}$	...	$\sqrt{2(2k^2-4k+5)(2k^2-2k+1)}$
$\frac{1}{\varepsilon(4,k)}$	$\sqrt{18}$					
$\frac{1}{\varepsilon(5,k)}$	$\sqrt{58}$					

Thanks